

# The NRQED lagrangian at order $1/M^4$

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## Abstract

The parity and time-reversal invariant effective lagrangian for a heavy fermion interacting with an abelian gauge field, i.e., NRQED, is constructed through order  $1/M^4$ . The implementation of Lorentz invariance in the effective theory becomes nontrivial at this order, and a complete solution for Wilson coefficient constraints is obtained. Matching conditions in the one-fermion sector are presented in terms of form factors and two-photon matrix elements of the nucleon. The extension of NRQED to describe interactions of the heavy fermion with a light fermion is introduced. Sample applications are discussed; these include the computation of nuclear structure effects in atomic bound states, the model-independent analysis of radiative corrections to low-energy lepton-nucleon scattering, and the study of static electromagnetic properties of nucleons.

# 1 Introduction

Nonrelativistic QED (NRQED) is an effective field theory [1] describing the interactions of nonrelativistic fermions with electromagnetic fields. NRQED interactions at order  $1/M^4$  have become relevant for describing radiative corrections to proton structure contributions in hydrogenic bound state spectroscopy [2, 3]. The NRQED lagrangian, properly constrained by Lorentz invariance, trivializes the derivation of low-energy theorems of Compton scattering [4] and automatically incorporates the intricate singularity structure of scattering amplitudes [5, 6]. It can be used to rigorously compute radiative corrections to low-energy lepton-nucleon scattering, and it also provides a model-independent framework within which to analyze static properties of nucleons, such as polarizabilities and generalized electromagnetic moments [7]. In this paper we derive a complete basis of operators and coefficient constraints through order  $1/M^4$  for the effective theory of nonrelativistic nucleons and leptons interacting with photons.<sup>1</sup>

An important formal issue first arises at order  $1/M^4$ . As recently discussed in [8], a “reparameterization invariance” ansatz for enforcing relativistic invariance breaks down at this order. We derive the correct implementation of Lorentz invariance constraints and the resulting Wilson coefficient relations (i.e., nonrenormalization theorems) through order  $1/M^4$ . For applications involving NRQED of the proton or neutron we perform leading order (in  $\alpha$ ) matching computations to relate the remaining undetermined coefficients to observables of lepton-nucleon scattering.

The remainder of the paper is structured as follows. In Section 2 we construct the NRQED lagrangian in the one-fermion sector through order  $1/M^4$ . This lagrangian describes the interaction of the proton with arbitrary background electromagnetic fields. Section 3 enforces constraints on the Wilson coefficients deriving from relativistic invariance, first by a variational calculation and then by an equivalent invariant operator construction. In Section 4 we perform matching calculations in the one-fermion sector, relating the undetermined Wilson coefficients to form factors and observables of (virtual) Compton scattering. In Section 5 we complete the basis of operators in the zero-fermion and two-fermion sectors, required to describe a proton interacting with a nonrelativistic lepton and dynamical photon. For applications to electron scattering ( $m_e, E \ll M$ ) we also consider the extension to the case of a relativistic lepton. Section 6 provides a brief discussion of applications to the computation of nuclear structure effects in atomic bound states; to the model-independent analysis of radiative corrections to low-energy lepton-nucleon scattering; and to the study of static electromagnetic properties of nucleons. Section 7 provides a concluding discussion.

## 2 Lagrangian

Consider the lagrangian for a heavy fermion coupled to an abelian gauge field. We enforce hermiticity and invariance under parity, time-reversal and rotational symmetries. We also perform field redefinitions to eliminate time derivatives acting on the fermion field (apart

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<sup>1</sup> For definiteness we will often refer to the heavy fermion  $\psi$  as the “nucleon”, and to a second fermion  $\chi$  in Section 5 as the “lepton”.

from the leading term); we refer to this choice as the “canonical form” of the heavy particle lagrangian. We thus find in the one-fermion sector through  $1/M^4$ ,

$$\begin{aligned}
\mathcal{L} = \psi^\dagger \Big\{ & iD_t + c_2 \frac{\mathbf{D}^2}{2M} + c_4 \frac{\mathbf{D}^4}{8M^3} + c_{Fg} \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2M} + c_{Dg} \frac{[\boldsymbol{\partial} \cdot \mathbf{E}]}{8M^2} + i c_{Sg} \frac{\boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8M^2} \\
& + c_{W1g} \frac{\{\mathbf{D}^2, \boldsymbol{\sigma} \cdot \mathbf{B}\}}{8M^3} - c_{W2g} \frac{D^i \boldsymbol{\sigma} \cdot \mathbf{B} D^i}{4M^3} + c_{p'p} g \frac{\boldsymbol{\sigma} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \mathbf{D}}{8M^3} \\
& + i c_{Mg} \frac{\{D^i, [\boldsymbol{\partial} \times \mathbf{B}]^i\}}{8M^3} + c_{A1} g^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8M^3} - c_{A2} g^2 \frac{\mathbf{E}^2}{16M^3} \\
& + c_{X1g} \frac{[\mathbf{D}^2, \mathbf{D} \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{D}]}{M^4} + c_{X2g} \frac{\{\mathbf{D}^2, [\boldsymbol{\partial} \cdot \mathbf{E}]\}}{M^4} + c_{X3g} \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial} \cdot \mathbf{E}]}{M^4} \\
& + i c_{X4g} g^2 \frac{\{D^i, [\mathbf{E} \times \mathbf{B}]^i\}}{M^4} + i c_{X5g} \frac{D^i \boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}) D^i}{M^4} + i c_{X6g} \frac{\epsilon^{ijk} \sigma^i D^j [\boldsymbol{\partial} \cdot \mathbf{E}] D^k}{M^4} \\
& + c_{X7g} g^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{B} [\boldsymbol{\partial} \cdot \mathbf{E}]}{M^4} + c_{X8g} g^2 \frac{[\mathbf{E} \cdot \boldsymbol{\partial} \boldsymbol{\sigma} \cdot \mathbf{B}]}{M^4} + c_{X9g} g^2 \frac{[\mathbf{B} \cdot \boldsymbol{\partial} \boldsymbol{\sigma} \cdot \mathbf{E}]}{M^4} + c_{X10g} g^2 \frac{[E^i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} B^i]}{M^4} \\
& + c_{X11g} g^2 \frac{[B^i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} E^i]}{M^4} + c_{X12g} g^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{E} \times [\partial_t \mathbf{E} - \boldsymbol{\partial} \times \mathbf{B}]}{M^4} + \mathcal{O}(1/M^5) \Big\} \psi. \tag{1}
\end{aligned}$$

We have defined  $D_t = \partial/\partial t + igZA^0$ ,  $D^i = \partial/\partial x^i - igZA^i$ , where  $-gZ = -e, +e$  or  $0$  for an electron, proton or neutron, respectively. We use the summation convention  $X^i Y^i \equiv \sum_{i=1}^3 X^i Y^i$ , and define  $[X, Y] \equiv XY - YX$ ,  $\{X, Y\} \equiv XY + YX$  to denote commutators and anticommutators as usual. Square brackets around quantities imply that derivatives act only within the bracket. Electric and magnetic fields are defined as usual by  $\mathbf{E} = -[\partial_t \mathbf{A}] - [\boldsymbol{\partial} A^0]$  and  $\mathbf{B} = [\boldsymbol{\partial} \times \mathbf{A}]$ . By the definition of  $\mathbf{E}$  and  $\mathbf{B}$ ,  $[\boldsymbol{\partial} \cdot \mathbf{B}] = 0$  and  $[\partial_t \mathbf{B} + \boldsymbol{\partial} \times \mathbf{E}] = 0$ .

The most general term in (1) is obtained by constructing all possible rotationally invariant, hermitian combinations of  $iD_t$ ,  $D^i$ ,  $E^i$ ,  $iB^i$ ,  $i\sigma^i$ , with parity requiring an even number of factors of  $D^i$  and  $E^i$ . The operators through  $1/M^3$  were previously introduced in [1, 9, 10]. Terms at  $1/M^4$  with two field strength factors  $E^i$  or  $B^i$  are straightforward to tabulate; note that we have used  $[\partial_t \mathbf{B}] = -[\boldsymbol{\partial} \times \mathbf{E}]$  and the assumption of canonical form to eliminate time derivatives of the magnetic field. Remaining terms at  $1/M^4$  involve one factor of electric field  $E^i$  and three spatial derivatives  $D^i$ . Spin-independent terms are straightforward to tabulate; the basis of operators parameterized by  $c_{X1}$ ,  $c_{X2}$ ,  $c_{X3}$  differs from other possible choices by terms involving commutators  $[D^i, D^j]$ , i.e., terms with two field strengths. For spin-dependent terms we use  $[\boldsymbol{\partial} \times \mathbf{E}] = -[\partial_t \mathbf{B}]$  and the assumption of canonical form to eliminate occurrences of  $[\boldsymbol{\partial} \times \mathbf{E}]$ . The three-vector identity,

$$D^i (\mathbf{E} \times \boldsymbol{\sigma})^j + (\boldsymbol{\sigma} \times \mathbf{D})^j E^i + \sigma^i (\mathbf{D} \times \mathbf{E})^j = \mathbf{D} \cdot \mathbf{E} \times \boldsymbol{\sigma} \delta^{ij}, \tag{2}$$

applied to remaining terms of the form  $\psi^\dagger D^i (\dots) D^j \psi$ , leaves the basis of operators parameterized by  $c_{X5}$ ,  $c_{X6}$ .

### 3 Relativistic invariance

The lagrangian (1) is invariant, by construction, under rotations and spacetime translations. The remaining constraints of relativity are enforced by demanding invariance under boosts. Here we derive these additional constraints, first by a variational calculation in Section 3.1, and then by an equivalent invariant operator construction in Section 3.2.

#### 3.1 Variational method

As detailed in [8], under infinitesimal boosts, with infinitesimal boost parameter  $\boldsymbol{\eta} = -\mathbf{q}/M$ , we may choose the heavy fermion to transform as

$$\psi \rightarrow e^{-i\mathbf{q}\cdot\mathbf{x}} \left\{ 1 + \frac{i\mathbf{q}\cdot\mathbf{D}}{2M^2} + \frac{i\mathbf{q}\cdot\mathbf{D}\mathbf{D}^2}{4M^4} - \frac{\boldsymbol{\sigma}\times\mathbf{q}\cdot\mathbf{D}}{4M^2} \left[ 1 + \frac{\mathbf{D}^2}{4M^2} \right] + \frac{ic_D g}{8M^3} \mathbf{q}\cdot\mathbf{E} + \frac{c_S g}{8M^3} \mathbf{q}\cdot\boldsymbol{\sigma}\times\mathbf{E} + \mathcal{O}(g/M^4, 1/M^6) + \dots \right\} \psi, \quad (3)$$

while derivatives and gauge fields transform as Lorentz vectors:

$$\mathbf{B} \rightarrow \mathbf{B} - \frac{1}{M} \mathbf{q} \times \mathbf{E}, \quad \mathbf{E} \rightarrow \mathbf{E} + \frac{1}{M} \mathbf{q} \times \mathbf{B}, \quad \mathbf{D} \rightarrow \mathbf{D} + \frac{1}{M} \mathbf{q} D_t, \quad D_t \rightarrow D_t + \frac{1}{M} \mathbf{q} \cdot \mathbf{D}. \quad (4)$$

Field strength-dependent terms in (3) have been chosen to maintain canonical form. Since we are interested in the canonical lagrangian through order  $1/M^4$ , we need not specify the explicit form of the order  $1/M^4$  field strength-dependent terms, denoted by  $\mathcal{O}(g/M^4)$ . A straightforward computation yields

$$\delta\mathcal{L} = \frac{1}{M} \delta\mathcal{L}_1 + \frac{1}{M^2} \delta\mathcal{L}_2 + \frac{1}{M^3} \delta\mathcal{L}_3 + \frac{1}{M^4} \delta\mathcal{L}_4 + \dots, \quad (5)$$

where

$$\begin{aligned} \delta\mathcal{L}_1 &= \psi^\dagger [(1 - c_2) i\mathbf{q}\cdot\mathbf{D}] \psi, \\ \delta\mathcal{L}_2 &= \psi^\dagger \left[ -\frac{1}{2} (1 - c_2) \{ \mathbf{q}\cdot\mathbf{D}, D_t \} + \frac{g}{4} (Z - 2c_F + c_S) \boldsymbol{\sigma} \times \mathbf{q}\cdot\mathbf{E} \right] \psi, \\ \delta\mathcal{L}_3 &= \psi^\dagger \left[ \frac{g}{8} \mathbf{q}\cdot[\boldsymbol{\partial} \times \mathbf{B}] (c_F - c_D + 2c_M) + \frac{i}{4} \{ \mathbf{q}\cdot\mathbf{D}, \mathbf{D}^2 \} (c_2 - c_4) \right. \\ &\quad \left. + \frac{ig}{8} \{ \mathbf{q}\cdot\mathbf{D}, \boldsymbol{\sigma}\cdot\mathbf{B} \} (c_2 Z + 2c_F - c_S - 2c_{W1} + 2c_{W2}) \right. \\ &\quad \left. + \frac{ig}{8} \{ \boldsymbol{\sigma}\cdot\mathbf{D}, \mathbf{q}\cdot\mathbf{B} \} (-c_2 Z + c_F - c_{p'p}) + \frac{ig}{8} \mathbf{q}\cdot\boldsymbol{\sigma} (\mathbf{D}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{D}) (-c_F + c_S - c_{p'p}) \right] \psi. \end{aligned} \quad (6)$$

From  $\delta\mathcal{L}_1$ ,  $\delta\mathcal{L}_2$  and  $\delta\mathcal{L}_3$ , we find [10, 3]<sup>2</sup>

$$c_2 = 1, \quad c_S = 2c_F - Z, \quad c_4 = 1, \quad 2c_M = c_D - c_F, \quad c_{W2} = c_{W1} - Z, \quad c_{p'p} = c_F - Z. \quad (7)$$

Employing the above relations, the variation  $\delta\mathcal{L}_4$  takes the form

$$\begin{aligned} \delta\mathcal{L}_4 = & \psi^\dagger \left[ \frac{ig}{8} [\mathbf{D}^2, \mathbf{q} \cdot \mathbf{E}] \left( \frac{5Z}{4} - c_F + c_D - 32c_{X1} \right) + \frac{ig}{8} \{ \mathbf{q} \cdot \mathbf{D}, [\boldsymbol{\partial} \cdot \mathbf{E}] \} \left( -\frac{Z}{4} + c_F - 16c_{X2} \right) \right. \\ & + \frac{g^2}{8} \mathbf{q} \cdot \mathbf{E} \times \mathbf{B} \left( \frac{Z^2}{2} + 2c_F(Z - c_F) - 2Zc_D + c_{A2} + 16c_{X4} \right) \\ & + \frac{g}{8} [\mathbf{q} \cdot \boldsymbol{\sigma} \times \boldsymbol{\partial} \boldsymbol{\partial} \cdot \mathbf{E}] \left( -Z + c_F - \frac{1}{4}c_D + c_{W1} + 8c_{X6} \right) \\ & \left. + \frac{g}{8} D^i (q^i (\mathbf{E} \times \boldsymbol{\sigma})^j + (\mathbf{E} \times \boldsymbol{\sigma})^i q^j + \boldsymbol{\sigma} \times \mathbf{q} \cdot \mathbf{E} \delta^{ij}) D^j \left( \frac{Z}{2} - 2c_F + 16c_{X5} \right) \right] \psi, \quad (8) \end{aligned}$$

where we have suppressed terms that are removed by field strength-dependent modifications of the boost generator, denoted by  $\mathcal{O}(g/M^4)$  in (3). We readily find,

$$\begin{aligned} 32c_{X1} &= \frac{5Z}{4} - c_F + c_D, \\ 32c_{X2} &= -\frac{Z}{2} + 2c_F, \\ 32c_{X4} &= -Z^2 - 4c_F(Z - c_F) + 4Zc_D - 2c_{A2}, \\ 32c_{X5} &= -Z + 4c_F, \\ 32c_{X6} &= 4(Z - c_F) + c_D - 4c_{W1}, \end{aligned} \quad (9)$$

while coefficients  $c_{X3}$  and  $c_{X7\dots X12}$  are not constrained by Lorentz invariance. We thus find that seven new quantities are required at order  $1/M^4$  to describe the proton's response to arbitrary background electromagnetic fields.

### 3.2 Invariant operators

An alternate method for enforcing Lorentz invariance is to construct the lagrangian from explicitly invariant operators. We summarize here the main points; for details see [8].

The basic building block in the construction is the field  $\Psi_v = \Gamma(v, iD)\psi_v$ , where  $\psi_v$  is a Dirac spinor field with  $\not{v}\psi_v = \psi_v$ . The matrix-valued operator  $\Gamma(v, iD)$  is determined such that under an infinitesimal boost  $\Lambda$ , where  $\Lambda^\mu_\nu v^\nu = v^\mu + q^\mu/M$ , the field  $\Psi_v$  has a simple transformation law:  $\Psi_v \rightarrow e^{iq \cdot x} \Psi_v$ . Noting that  $e^{-iq \cdot x} (iD^\mu + Mv^\mu + q^\mu) e^{iq \cdot x} = iD^\mu + Mv^\mu$ , we may thus build invariant bilinears from contractions of polynomials of  $\gamma^\mu$  and  $\mathcal{V}^\mu \equiv v^\mu + iD^\mu/M$ , between  $\bar{\Psi}_v$  and  $\Psi_v$ .

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<sup>2</sup> As noted in [3], we find the opposite sign in the relation for  $c_M$  in (7) compared to [10].

The function  $\Gamma(v, iD)$  is a solution to the invariance equation,

$$\Gamma(v + q/M, iD - q)\Lambda^{-1}W(\Lambda, iD + Mv) = \Gamma(v, iD), \quad (10)$$

where  $W(\Lambda, p)$  is an element of the little group for timelike invariant vector  $v^\mu$ , following from the theory of induced representations of the Lorentz group. Up to the relevant order for determining the  $1/M^4$  lagrangian we have [8]

$$\begin{aligned} \Gamma = 1 + \frac{i\mathcal{D}_\perp}{2M} + \frac{1}{M^2} \left\{ -\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\mathcal{D}_\perp iv \cdot D \right\} + \frac{1}{M^3} \left\{ \frac{1}{4}(iD_\perp)^2 iv \cdot D \right. \\ \left. + \frac{i\mathcal{D}_\perp}{2} \left[ -\frac{3}{8}(iD_\perp)^2 + (iv \cdot D)^2 \right] + \frac{gZ}{8}F_{\mu\nu}v^\mu D_\perp^\nu + \frac{gZ}{16}\sigma_\perp^{\mu\nu}F_{\mu\nu}i\mathcal{D}_\perp \right\} + \dots, \end{aligned} \quad (11)$$

where we have defined  $D_\perp^\mu \equiv D^\mu - v^\mu v \cdot D$ , and for abelian gauge fields  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ . Note that the last two terms of (11) are absent in the ansatz for reparameterization invariance given in [11], leading to incorrect Lorentz invariance constraints at  $1/M^4$  and beyond. This subtlety is explained in [8].

A complete basis of invariant bilinears required through order  $1/M^4$  is

$$\begin{aligned} \mathcal{L} = \bar{\Psi}_v \left\{ M(\mathcal{V} - 1) - a_F g \frac{\sigma^{\mu\nu} F_{\mu\nu}}{4M} + ia_D g \frac{\{\mathcal{V}_\mu, [M\mathcal{V}_\nu, F^{\mu\nu}]\}}{16M^2} - a_{W1} g \frac{[M\mathcal{V}^\alpha, [M\mathcal{V}_\alpha, \sigma^{\mu\nu} F_{\mu\nu}]]}{16M^3} \right. \\ \left. + a_{A1} g^2 \frac{F_{\mu\nu} F^{\mu\nu}}{16M^3} + a_{A2} g^2 \frac{\mathcal{V}_\alpha F^{\mu\alpha} F_{\mu\beta} \mathcal{V}^\beta}{16M^3} \right\} \Psi_v + a_{X3} \mathcal{B}_{X3} + \sum_{i=7}^{12} a_{Xi} \mathcal{B}_{Xi}. \end{aligned} \quad (12)$$

The bilinears  $\mathcal{B}_{Xi}$  for  $i = 3, 7 \dots 12$  are chosen to reduce to the respective operators multiplying  $c_{Xi}$  in (1) upon setting  $v^\mu = (1, 0, 0, 0)$  and neglecting  $1/M$  suppressed corrections. Since we are concerned only with the lagrangian through order  $1/M^4$  we do not specify an explicit choice for these  $\mathcal{B}_{Xi}$ . A computation shows that the field redefinition to recover canonical form is

$$\begin{aligned} \psi_v = \left\{ 1 + \frac{1}{4M^2}(iD_\perp)^2 \left( 1 - \frac{iv \cdot D}{M} \right) - \frac{gZ}{16M^2}\sigma_\perp^{\mu\nu}F_{\mu\nu} - \frac{gZ}{4M^3}D_\perp^\mu v^\alpha F_{\alpha\mu} + \frac{igZ}{4M^3}\sigma_{\mu\nu}D_\perp^\mu v_\alpha F^{\alpha\nu} \right. \\ \left. - \frac{gZ}{8M^3}v^\alpha F_{\alpha\mu}D_\perp^\mu + \frac{ga_F}{4M^3}[-D_\perp^\mu v^\alpha F_{\alpha\mu} + i\sigma_{\mu\nu}D_\perp^\mu v_\alpha F^{\alpha\nu}] - \frac{ga_D}{8M^3}v^\alpha F_{\alpha\mu}D_\perp^\mu \right. \\ \left. + \frac{iga_{W1}}{8M^3}\sigma_{\mu\nu}[D_\perp^\mu, v_\alpha F^{\alpha\nu}] \right\} \psi'_v. \end{aligned} \quad (13)$$

Upon setting  $v^\mu = (1, 0, 0, 0)$ , the resulting lagrangian, expressed in terms of  $\psi'_v$ , is identical to the previous result (1) with constraints (7) and (9).

## 4 Matching: one-fermion sector

In contrast to NRQED for the electron or other fundamental fermions, the matching for a composite particle such as the proton cannot be performed perturbatively. We instead must

appeal to nonperturbative (e.g. lattice) methods, or to experimental measurements. This section relates the matching conditions in the one-fermion sector to standard form factors and two-photon matrix elements of the nucleon.

## 4.1 One-photon matching

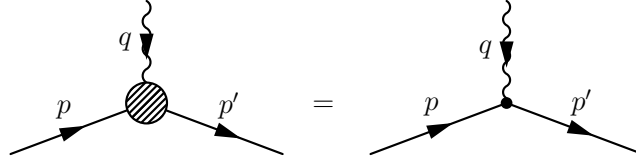


Figure 1: Tree level matching of the one-photon amplitude in the full theory and NRQED. The black dot in the diagram on the right-hand side represents single-photon NRQED vertices.

Consider first the operators contributing to the one-photon matrix element of the nucleon. The matching is performed in terms of standard invariant form factors,

$$\langle N(p') | J_\mu^{\text{em}} | N(p) \rangle = \bar{u}(p') \Gamma_\mu(q) u(p), \quad \Gamma_\mu(q) \equiv \gamma_\mu F_1^N(q^2) + \frac{i\sigma_{\mu\nu}}{2M_N} F_2^N(q^2) q^\nu, \quad (14)$$

where  $q = p' - p$  and  $N$  denotes a proton or neutron; we suppress the superscript  $N$  in the following. Equating the effective theory with the full theory<sup>3</sup>, we find (cf. Fig. 1)

$$\begin{aligned} c_F &= \bar{F}_1 + \bar{F}_2 \equiv Z + a_N + \mathcal{O}(\alpha), \\ c_D &= \bar{F}_1 + 2\bar{F}_2 + 8\bar{F}'_1 \equiv Z + \frac{4}{3}M^2(r_E^N)^2 + \mathcal{O}(\alpha), \\ c_{W1} &= \bar{F}_1 + \frac{1}{2}\bar{F}_2 + 4\bar{F}'_1 + 4\bar{F}'_2, \\ c_{X3} &= \frac{1}{8}\bar{F}'_1 + \frac{1}{4}\bar{F}'_2 + \frac{1}{2}\bar{F}''_1, \end{aligned} \quad (15)$$

where  $Z$  denotes the electric charge,  $a_N$  is the anomalous magnetic moment of the nucleon, and  $r_E^N$  is the nucleon charge radius. We have introduced dimensionless barred quantities to denote derivatives with respect to  $q^2/M^2$  at  $q^2 = 0$ :  $\bar{F}_1 \equiv F_1(0) = Z$ ,  $\bar{F}_2 \equiv F_2(0) = a_N$ ,  $\bar{F}'_i \equiv M^2 F'_i(0)$ , etc. The new quantity  $\bar{F}''_1$  appears at  $1/M^4$ . Expressions for other Wilson coefficients through  $1/M^3$  in terms of form factors can be found using (7). At  $1/M^4$ , we also find

$$c_{X1} = \frac{5}{128}\bar{F}_1 + \frac{1}{32}\bar{F}_2 + \frac{1}{4}\bar{F}'_1,$$

<sup>3</sup> The nonrelativistic normalization of states in NRQED is obtained using  $\bar{u}(p)u(p) = M/E_{\mathbf{p}}$  in (14).

$$\begin{aligned}
c_{X2} &= \frac{3}{64}\bar{F}_1 + \frac{1}{16}\bar{F}_2, \\
c_{X5} &= \frac{3}{32}\bar{F}_1 + \frac{1}{8}\bar{F}_2, \\
c_{X6} &= -\frac{3}{32}\bar{F}_1 - \frac{1}{8}\bar{F}_2 - \frac{1}{4}\bar{F}'_1 - \frac{1}{2}\bar{F}'_2,
\end{aligned} \tag{16}$$

and it is readily verified that these expressions satisfy the constraints (9). In the presence of radiative corrections, the form factors on the right hand sides of (15) and (16) should be interpreted in an appropriate infrared regularization scheme; alternatively, the matching may be performed with infrared finite observables. The corresponding infrared subtractions and ultraviolet renormalizations must be performed to obtain the Wilson coefficients including radiative corrections.<sup>4</sup>

## 4.2 Two-photon matching

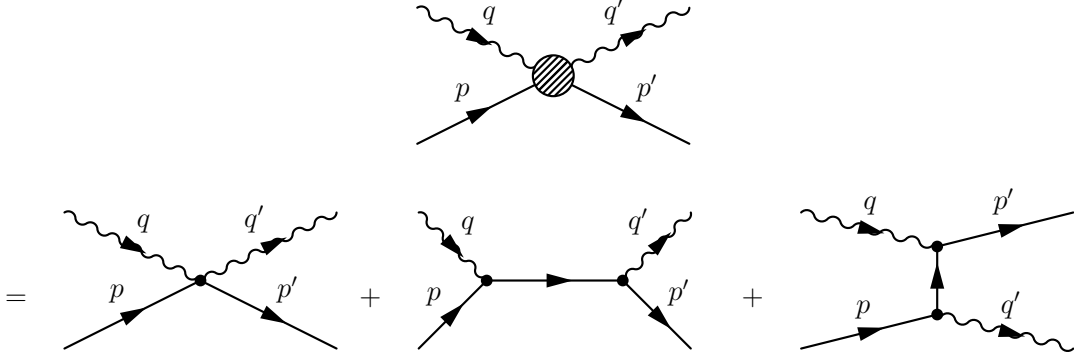


Figure 2: Tree-level matching of the Compton scattering amplitude in the full theory and NRQED. The black vertices in the diagrams on the right-hand side represent NRQED vertices.

The Compton scattering process,  $\gamma^*(q)N(p) \rightarrow \gamma(q')N(p')$ , with one virtual and one real photon is sufficient to determine the remaining coefficients in the  $1/M^4$  NRQED lagrangian. Consider the low-energy expansion of the virtual Compton scattering amplitude as depicted in Fig. 2. Let us define a conventional separation

$$\mathcal{M}^{\mu\nu} \equiv \mathcal{M}_{\text{Born}}^{\mu\nu} + \mathcal{M}_{\text{non-Born}}^{\mu\nu}, \tag{17}$$

by declaring that “Born” terms are defined by the “Sticking In Form Factors” (SIFF) prescription using the form factors of (14). Explicitly,

$$\mathcal{M}_{\text{Born}}^{\mu\nu} \equiv -g^2 \bar{u}(p') \left\{ \Gamma^\nu(-q') \frac{1}{\not{p} + \not{q} - M} \Gamma^\mu(q) + \Gamma^\mu(q) \frac{1}{\not{p} - \not{q}' - M} \Gamma^\nu(-q') \right\} u(p). \tag{18}$$

<sup>4</sup>The expressions on the right hand side of (15) and (16) correspond to those referred to as  $c_i^{\text{QED}}$  in [9]. The renormalization procedure in dimensional regularization is described in [10].



This convention ensures that  $q_\mu \mathcal{M}_{\text{Born}}^{\mu\nu} = q'_\nu \mathcal{M}_{\text{Born}}^{\mu\nu} = 0$ , so that the same Ward identities,  $q_\mu \mathcal{M}_{\text{non-Born}}^{\mu\nu} = q'_\nu \mathcal{M}_{\text{non-Born}}^{\mu\nu} = 0$ , may be applied to constrain  $\mathcal{M}_{\text{non-Born}}^{\mu\nu}$ . We adopt a convenient basis for  $\mathcal{M}_{\text{non-Born}}^{\mu\nu}$  as given by Drechsel et al (Eq. (A10) of [12]),

$$\mathcal{M}_{\text{non-Born}}^{\mu\nu} = \sum_{i=1}^{12} f_i(q^2, q \cdot q', q \cdot p) \rho_i^{\mu\nu}. \quad (19)$$

Subtraction of the Born terms ensures that the residual contributions  $f_i(q^2, q \cdot q', q \cdot p)$  are free of kinematic singularities, and may thus be Taylor expanded at small photon energy. Employing  $q \cdot p \sim \omega$ ,  $q^2 \sim \omega^2$ ,  $q \cdot q' \sim \omega^2$ , to the relevant order, we require

$$\begin{aligned} f_i(q^2, q \cdot q', q \cdot p) &\equiv f_{i,0} + \mathcal{O}(\omega^2), \quad i = 1, 2, 5, 6, 11, 12, \\ f_i(q^2, q \cdot q', q \cdot p) &\equiv f_{i,1} q \cdot p + \mathcal{O}(\omega^3), \quad i = 4, 10, \end{aligned} \quad (20)$$

where we employ the notation of [13].

The matching can be performed in arbitrary reference frame, with the results,<sup>5</sup>

$$\begin{aligned} c_{A1} &= \bar{F}_1^2 + 4\bar{f}_{1,0} \equiv Z^2 + \frac{4M^3}{\alpha} \beta_M + \mathcal{O}(\alpha), \\ c_{A2} &= 4\bar{F}_1 \bar{F}_2 + 2\bar{F}_2^2 + 16\bar{F}_1 \bar{F}'_1 + 32\bar{f}_{2,0} \equiv 2a_N^2 + \frac{8Z}{3} M^2 (r_E^N)^2 - \frac{8M^3}{\alpha} (\alpha_E + \beta_M) + \mathcal{O}(\alpha), \\ c_{X7} &= \frac{1}{4} \bar{F}_1 \bar{F}'_1 + \frac{1}{2} \bar{F}_1 \bar{F}'_2 - \frac{1}{2} \bar{f}_{5,0} - 2\bar{f}_{11,0} - 2\bar{f}_{12,0}, \\ c_{X8} &= -\frac{9}{32} \bar{F}_1^2 - \frac{3}{8} \bar{F}_1 \bar{F}_2 - \frac{1}{16} \bar{F}_2^2 - \left( \bar{F}_1 + \frac{1}{2} \bar{F}_2 \right) \bar{F}'_1 - \frac{1}{2} \bar{F}_1 \bar{F}'_2 - 4\bar{f}_{4,1} + 4\bar{f}_{6,0} - 2\bar{f}_{10,1}, \\ c_{X9} &= \frac{5}{64} \bar{F}_1^2 + \frac{3}{16} \bar{F}_1 \bar{F}_2 + \frac{1}{8} \bar{F}_2^2 + \frac{1}{2} (\bar{F}_1 + \bar{F}_2) (\bar{F}'_1 + \bar{F}'_2) - 2\bar{f}_{10,1}, \\ c_{X10} &= \frac{7}{32} \bar{F}_1^2 + \frac{3}{8} \bar{F}_1 \bar{F}_2 + \frac{1}{16} \bar{F}_2^2 + 4\bar{f}_{4,1} + 2\bar{f}_{10,1} + 2\bar{f}_{11,0}, \\ c_{X11} &= \frac{7}{64} \bar{F}_1^2 + \frac{3}{16} \bar{F}_1 \bar{F}_2 - \frac{1}{2} (\bar{F}_1 + \bar{F}_2) (\bar{F}'_1 + \bar{F}'_2) + 2\bar{f}_{10,1} + 2\bar{f}_{11,0}, \\ c_{X12} &= -\frac{1}{16} \bar{F}_1^2 - \frac{1}{8} \bar{F}_1 \bar{F}_2 - \frac{1}{16} \bar{F}_2^2 - \frac{1}{2} (\bar{F}_1 + \bar{F}_2) \bar{F}'_1 - \frac{1}{2} \bar{F}_1 \bar{F}'_2 + 4\bar{f}_{4,1} + \frac{1}{2} \bar{f}_{5,0} + 2\bar{f}_{10,1} + 2\bar{f}_{11,0}, \end{aligned} \quad (21)$$

where the dimensionless form factor derivatives,  $\bar{F}_i^{(n)}$ , have been introduced after (15) and we have similarly defined dimensionless quantities  $\bar{f}_1 = M^3 f_1$ ,  $\bar{f}_2 = M^5 f_2$ ,  $\bar{f}_4 = M^4 f_4$ ,  $\bar{f}_5 = M^4 f_5$ ,  $\bar{f}_6 = M^5 f_6$ ,  $\bar{f}_{10} = M^3 f_{10}$ ,  $\bar{f}_{11} = M^4 f_{11}$ ,  $\bar{f}_{12} = M^5 f_{12}$ . We have then used the expansion of the amplitudes  $f_i$  given in (20), with dimensionless quantities  $\bar{f}_i = \bar{f}_{i,0} + \mathcal{O}(\omega^2)$  ( $i = 1, 2, 5, 6, 11, 12$ )

<sup>5</sup>We have performed the computation both in the laboratory frame,  $\mathbf{p} = 0$ , and in the center of mass frame,  $\mathbf{q} + \mathbf{p} = \mathbf{0}$ , by matching both the full theory and the effective theory to the twelve independent spin structures for the virtual Compton scattering process.

and  $\bar{f}_i = \bar{f}_{i,1} q \cdot p / M^2 + \mathcal{O}(\omega^3)$  ( $i = 4, 10$ ). Note that six new phenomenological parameters are required at  $1/M^4$  to match  $c_{X7\dots X12}$ . An expression for  $c_{X4}$  in terms of scattering observables can be found using (9). The expressions (21) can be translated to a multitude of other conventions for the observables of Compton scattering.<sup>6</sup>

## 5 Pure photon and four-fermion operators

So far our analysis has focused on the one-fermion sector. We have derived the form of the lagrangian appropriate, e.g., to a proton in a background electromagnetic field. Let us consider the complete QED theory including dynamical photon, as well as a lepton (electron or muon) field. The case of a nonrelativistic lepton is appropriate to bound state hydrogen studies, or very low-energy lepton-nucleon (e.g. muon-proton) scattering, where  $E \ll M_\chi, M$ . We first consider this case, constructing the operator basis, deriving coefficient relations and identifying redundant operators. We then turn to a brief discussion of the case of a relativistic lepton, appropriate to e.g. low-energy electron-proton scattering with  $m_\ell, E \ll M$ .

### 5.1 Pure photon operators

The pure gauge sector for NRQED is the well known Euler-Heisenberg lagrangian. Enforcing parity and time reversal symmetry and neglecting total derivatives we find

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_{V2}\frac{F_{\mu\nu}[\partial^2 F^{\mu\nu}]}{M^2} + c_{V4}\frac{F_{\mu\nu}[\partial^4 F^{\mu\nu}]}{M^4} + c_{E1}g^2\frac{(F_{\mu\nu}F^{\mu\nu})^2}{M^4} + c_{E2}g^2\frac{F_\nu^\mu F_\rho^\nu F_\sigma^\rho F_\mu^\sigma}{M^4} + \dots \quad (22)$$

The coefficients  $c_{V2}$  and  $c_{V4}$  may be set to zero through field redefinitions on  $A^\mu$ , as discussed in Section 5.3 below.

### 5.2 Four-fermion operators

Consider four-fermion operators relevant for processes in the one-nucleon, one-lepton sector. We enforce hermiticity and invariance under parity, time-reversal and rotational symmetries. We use the notation  $\overleftarrow{D}$  for a covariant derivative acting to the left,  $X\overleftarrow{D}^i \equiv [\partial^i X] + igZX A^i$ , and define  $D_+ \equiv D + \overleftarrow{D}$ ,  $D_- \equiv D - \overleftarrow{D}$ . Having performed field redefinitions to eliminate operators with time derivatives acting on heavy fermions, the lagrangian in this sector, through  $1/M^4$ , is

$$\begin{aligned} \mathcal{L}_{\psi\chi} = & \frac{d_1}{M^2}\psi^\dagger\sigma^i\psi\chi^\dagger\sigma^i\chi + \frac{d_2}{M^2}\psi^\dagger\psi\chi^\dagger\chi + \frac{d_3}{M^4}\psi^\dagger D_+^i\psi\chi^\dagger D_+^i\chi + \frac{d_4}{M^4}\psi^\dagger D_-^i\psi\chi^\dagger D_-^i\chi \\ & + \frac{d_5}{M^4}\psi^\dagger(\mathbf{D}^2 + \overleftarrow{\mathbf{D}}^2)\psi\chi^\dagger\chi + \frac{d_6}{M^4}\psi^\dagger\psi\chi^\dagger(\mathbf{D}^2 + \overleftarrow{\mathbf{D}}^2)\chi \end{aligned}$$

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<sup>6</sup> For example, Compton scattering of real photons determines all coefficients, apart from  $c_{X7}$  and  $c_{X12}$ , in terms of the conventional electric and magnetic polarizabilities  $\alpha_E, \beta_M$  and Ragusa's [4] spin polarizabilities  $\gamma_i$ :  $M^3\alpha_E/\alpha = -\bar{f}_{1,0} - 4\bar{f}_{2,0}$ ,  $M^3\beta_M/\alpha = \bar{f}_{1,0}$ ,  $M^4\gamma_1/\alpha = -8\bar{f}_{4,1} - 4\bar{f}_{10,1} - 4\bar{f}_{11,0}$ ,  $M^4\gamma_2/\alpha = 4\bar{f}_{10,1}$ ,  $M^4\gamma_3/\alpha = 4\bar{f}_{6,0} + 2\bar{f}_{11,0}$ ,  $M^4\gamma_4/\alpha = -4\bar{f}_{10,1} - 2\bar{f}_{11,0}$ , where  $\alpha$  is the fine structure constant. Compare with Eq. (28) of [13].

$$\begin{aligned}
& + \frac{gd_7}{M^4} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B} \psi \chi^\dagger \chi + \frac{id_8}{M^4} \epsilon^{ijk} \psi^\dagger \sigma^i D_-^j \psi \chi^\dagger D_+^k \chi + \frac{id_9}{M^4} \epsilon^{ijk} \psi^\dagger \sigma^i D_+^j \psi \chi^\dagger D_-^k \chi \\
& + \frac{gd_{10}}{M^4} \psi^\dagger \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B} \chi + \frac{id_{11}}{M^4} \epsilon^{ijk} \psi^\dagger D_+^k \psi \chi^\dagger \sigma^i D_-^j \chi + \frac{id_{12}}{M^4} \epsilon^{ijk} \psi^\dagger D_-^k \psi \chi^\dagger \sigma^i D_+^j \chi \\
& + \frac{d_{13}}{M^4} \psi^\dagger \sigma^i D_+^j \psi \chi^\dagger \sigma^i D_+^j \chi + \frac{d_{14}}{M^4} \psi^\dagger \sigma^i D_-^j \psi \chi^\dagger \sigma^i D_-^j \chi + \frac{d_{15}}{M^4} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_+ \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_+ \chi \\
& + \frac{d_{16}}{M^4} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_- \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_- \chi + \frac{d_{17}}{M^4} \psi^\dagger \sigma^i D_-^j \psi \chi^\dagger \sigma^j D_-^i \chi \\
& + \frac{d_{18}}{M^4} \psi^\dagger \sigma^i (\mathbf{D}^2 + \overleftarrow{\mathbf{D}}^2) \psi \chi^\dagger \sigma^i \chi + \frac{d_{19}}{M^4} \psi^\dagger \sigma^i (D^i D^j + \overleftarrow{D}^j \overleftarrow{D}^i) \psi \chi^\dagger \sigma^j \chi \\
& + \frac{d_{20}}{M^4} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^i (\mathbf{D}^2 + \overleftarrow{\mathbf{D}}^2) \chi + \frac{d_{21}}{M^4} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^j (D^i D^j + \overleftarrow{D}^j \overleftarrow{D}^i) \chi. \tag{23}
\end{aligned}$$

Here  $\chi$  is the nonrelativistic lepton field with mass  $M_\chi$  and for notational simplicity we write all operators in terms of the common mass scale  $M$ .<sup>7</sup> Covariant derivatives appearing within a fermion bilinear in (23) are understood to act only on fields in that bilinear. The heavy field  $\psi$  transforms under boosts as in (3). Recalling that  $\mathbf{q}$  in (3) is related to the mass-independent infinitesimal boost parameter by  $\boldsymbol{\eta} = -\mathbf{q}/M$ , the transformation law for  $\chi$  is obtained by the replacement  $M \rightarrow rM$  and  $q \rightarrow rq$ , where we define  $r \equiv M_\chi/M$ . We thus find

$$\begin{aligned}
\delta \mathcal{L}_\psi = \frac{1}{M^4} & \left\{ \psi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \psi \chi^\dagger \chi \left[ \frac{d_2}{2} - 2rd_4 - 2d_5 \right] + \psi^\dagger \psi \chi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \chi \left[ \frac{d_2}{2r} - 2d_4 - 2rd_6 \right] \right. \\
& + \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \times \mathbf{D}_+ \psi \chi^\dagger \chi \left[ -\frac{d_2}{4} + \frac{d_1}{4r} - 2d_8 - 2rd_9 \right] + \psi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \sigma^i \psi \chi^\dagger \sigma^i \chi \left[ \frac{d_1}{2} - 2rd_{14} - 2d_{18} \right] \\
& + \psi^\dagger \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \times \mathbf{D}_+ \chi \left[ -\frac{d_2}{4r} + \frac{d_1}{4} - 2rd_{11} - 2d_{12} \right] + \psi^\dagger \sigma^i \psi \chi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \sigma^i \chi \left[ \frac{d_1}{2r} - 2d_{14} - 2rd_{20} \right] \\
& + \psi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{D}_- \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \chi \left[ \frac{d_1}{4} - 2rd_{16} - d_{19} \right] + \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \psi \chi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{D}_- \chi \left[ \frac{d_1}{4r} - 2d_{16} - rd_{21} \right] \\
& + \psi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{q} D_-^i \psi \chi^\dagger \sigma^i \chi \left[ -\frac{d_1}{4} - 2rd_{17} - d_{19} \right] + \psi^\dagger \sigma^i \psi \chi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{q} D_-^i \chi \left[ -\frac{d_1}{4r} - 2d_{17} - rd_{21} \right] \Big\} \\
& + \mathcal{O}(1/M^5). \tag{24}
\end{aligned}$$

This enforces the relations

$$\begin{aligned}
rd_4 + d_5 &= \frac{d_2}{4}, \quad d_5 = r^2 d_6, \quad 8r(d_8 + rd_9) = -rd_2 + d_1, \quad 8r(rd_{11} + d_{12}) = -d_2 + rd_1, \\
rd_{14} + d_{18} &= \frac{d_1}{4}, \quad d_{18} = r^2 d_{20}, \quad 2rd_{16} + d_{19} = \frac{d_1}{4}, \quad r(d_{16} + d_{17}) + d_{19} = 0, \quad d_{19} = r^2 d_{21}, \tag{25}
\end{aligned}$$

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<sup>7</sup> Note that the coefficients  $d_{1,2}$  in (23) are related to those of Caswell and Lepage [1] by a factor  $M_\chi/M$ .

implying a total of twelve independent four-fermion operators through  $1/M^4$ , including two at order  $1/M^2$ . By performing field redefinitions on the gauge field  $A^\mu$ , some of these four-fermion operators are found to mix with one-heavy particle sector operators, as discussed in Section 5.3 below.

The lagrangian (23), with constraints (25), applies to the case of distinct heavy particles represented by  $\psi$ ,  $\chi$ , with arbitrary mass ratio  $M_\chi/M$ . For certain applications, e.g. positronium or heavy quarkonium bound states, the fields  $\psi$  and  $\chi$  can be taken to represent particle-antiparticle pairs with  $r = M_\chi/M = 1$ . Charge conjugation symmetry is then implemented by enforcing invariance under  $\psi \leftrightarrow \chi$ , thus reducing the basis of operators. This case has been investigated for QCD through  $\mathcal{O}(1/M^4)$  by Brambilla et al. [14]. We find that our basis of four-fermion operators (23) and constraints (25) are equivalent to those found in Ref. [14] for this special case.<sup>8</sup>

### 5.3 Field redefinitions and redundant operators

With a dynamical photon field, we may perform field redefinitions that maintain reality and gauge, parity, time reversal and rotational symmetries. In order to avoid upsetting the previously determined coefficient relations, we must also maintain the transformation law for  $A^\mu$  as a four-vector under Lorentz transformations, i.e.,

$$A^0 \rightarrow A^0 - \frac{1}{M} \mathbf{q} \cdot \mathbf{A}, \quad \mathbf{A} \rightarrow \mathbf{A} - \frac{1}{M} \mathbf{q} A^0. \quad (26)$$

Let us write

$$A_\mu = A'_\mu + \Delta_\gamma A_\mu + \Delta_\psi A_\mu + \Delta_\chi A_\mu + \dots \quad (27)$$

For the pure gauge field terms the most general expression is

$$\Delta_\gamma A^\mu = a_{\gamma 1} \frac{\partial_\nu F^{\nu\mu}}{M^2} + a_{\gamma 2} \frac{\partial^2 \partial_\nu F^{\nu\mu}}{M^4} + \mathcal{O}(1/M^6). \quad (28)$$

Terms involving the heavy fermion  $\psi$  take the form

$$\begin{aligned} \frac{\Delta_\psi A^\mu}{g} = & \tilde{a}_{\psi 1} \frac{\bar{\Psi}_v \gamma^\mu \Psi_v}{M^2} + \tilde{a}_{\psi 2} \frac{\partial_\alpha (\bar{\Psi}_v \sigma^{\alpha\mu} \Psi_v)}{M^3} + \tilde{a}_{\psi 3} g \frac{\bar{\Psi}_v \{\gamma^\mu, \sigma^{\alpha\beta} F_{\alpha\beta}\} \Psi_v}{M^4} + \tilde{a}_{\psi 4} \frac{\partial^2 (\bar{\Psi}_v \gamma^\mu \Psi_v)}{M^4} \\ & + \tilde{a}_{\psi 5} g \frac{\bar{\Psi}_v \sigma^{\mu\alpha} \{\mathcal{V}^\beta, F_{\alpha\beta}\} \Psi_v}{M^4} + \mathcal{O}(1/M^5), \end{aligned} \quad (29)$$

where we have employed the invariant operator formalism of Section 3.2. In particular,  $\Psi_v = \Gamma \psi_v$  with  $\Gamma$  from (11) and  $\psi_v$  from (13), expressed in terms of the field  $\psi'_v \equiv \psi$  with canonical lagrangian (1). As an alternative to the invariant operator formalism employed in (29) we may expand  $\Delta_\psi A^0$  and  $\Delta_\psi \mathbf{A}$  in a series of rotationally invariant operators with arbitrary

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<sup>8</sup>The difference between abelian and nonabelian gauge fields is trivial for four-fermion operators through this order.

coefficients, and subsequently constrain these coefficients using (26). The result is equivalent to (29), with five free parameters through  $\mathcal{O}(1/M^4)$ ,

$$\begin{aligned}\frac{\Delta_\psi A^0}{g} &= a_{\psi 1} \frac{\psi^\dagger \psi}{M^2} + a_{\psi 2} \frac{\partial^2(\psi^\dagger \psi)}{M^4} - i \left( \frac{a_{\psi 1}}{4} - a_{\psi 4} \right) \frac{\psi^\dagger \boldsymbol{\sigma} \cdot \overleftarrow{\mathbf{D}} \times \mathbf{D} \psi}{M^4} + a_{\psi 3} g \frac{\psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B} \psi}{M^4} + \mathcal{O}(1/M^5), \\ \frac{\Delta_\psi \mathbf{A}}{g} &= -a_{\psi 1} \frac{\psi^\dagger i \mathbf{D}_- \psi}{2M^3} + a_{\psi 4} \frac{\boldsymbol{\partial} \times (\psi^\dagger \boldsymbol{\sigma} \psi)}{M^3} + a_{\psi 5} g \frac{\psi^\dagger \boldsymbol{\sigma} \times \mathbf{E} \psi}{M^4} + \mathcal{O}(1/M^5).\end{aligned}\quad (30)$$

The expansion of  $\Delta_\chi A^\mu$  is obtained from (30) with the replacements  $\psi \rightarrow \chi$ ,  $M \rightarrow M_\chi$ ,  $Z \rightarrow Z_\chi$  and  $a_{\psi i} \rightarrow a_{\chi i}$ . In terms of the field  $A'_\mu$  in (27), we find in the pure photon sector,

$$\delta c_{V2} = -\frac{1}{2}a_{\gamma 1}, \quad \delta c_{V4} = -\frac{1}{2}a_{\gamma 2} - \frac{1}{4}a_{\gamma 1}^2 + 2a_{\gamma 1}c_{V2}, \quad (31)$$

while for the  $\psi$  sector,

$$\begin{aligned}\delta c_D &= -8Za_{\gamma 1} + 8a_{\psi 1}, \quad \delta c_{W1} = -4c_F a_{\gamma 1} + 8a_{\psi 4}, \quad \delta c_{A2} = -16Z^2 a_{\gamma 1} + 16Za_{\psi 1}, \\ \delta c_{X3} &= -\frac{c_D a_{\gamma 1}}{8} + Za_{\gamma 2} - a_{\gamma 1}a_{\psi 1} + 4c_{V2}a_{\psi 1} + a_{\psi 2}, \quad \delta c_{X7} = -\frac{c_S Z a_{\gamma 1}}{4} + a_{\psi 3}, \\ \delta c_{X8} &= c_F Z a_{\gamma 1} - \frac{c_F a_{\psi 1}}{2} - Za_{\psi 4}, \quad \delta c_{X9} = -\frac{c_F^2 a_{\gamma 1}}{2} + c_F a_{\psi 4}, \quad \delta c_{X11} = \frac{c_F^2 a_{\gamma 1}}{2} - c_F a_{\psi 4}, \\ \delta c_{X12} &= \frac{c_S Z a_{\gamma 1}}{2} + a_{\psi 5}.\end{aligned}\quad (32)$$

Similar relations hold for the Wilson coefficients  $c_i^{(\chi)}$  in the  $\chi$  lagrangian, defined as in (1), with  $\psi \rightarrow \chi$ ,  $Z \rightarrow Z_\chi$ ,  $M \rightarrow M_\chi$ ,  $c_i \rightarrow c_i^{(\chi)}$ . Finally, for the four-fermion operator coefficients,

$$\begin{aligned}\frac{\delta d_2}{g^2} &= -Z_\chi a_{\psi 1} - \frac{Z a_{\chi 1}}{r^2}, \quad \frac{\delta d_3}{g^2} = \frac{c_D^{(\chi)} a_{\psi 1}}{8r^2} + \frac{c_D a_{\chi 1}}{8r^2} + Z_\chi a_{\psi 2} + \frac{Z a_{\chi 2}}{r^4}, \\ \frac{\delta d_4}{g^2} &= -\frac{Z_\chi a_{\psi 1}}{4r} - \frac{Z a_{\chi 1}}{4r^3}, \quad \frac{\delta d_7}{g^2} = -\frac{Z Z_\chi}{4} (a_{\psi 1} - 4a_{\psi 4}) - Z_\chi a_{\psi 3}, \\ \frac{\delta d_8}{g^2} &= \frac{Z_\chi}{8} (a_{\psi 1} - 4a_{\psi 4}) - \frac{c_S a_{\chi 1}}{8r^2}, \quad \frac{\delta d_{10}}{g^2} = -\frac{Z Z_\chi}{4r^4} (a_{\chi 1} - 4a_{\chi 4}) - \frac{Z a_{\chi 3}}{r^4}, \\ \frac{\delta d_{11}}{g^2} &= -\frac{c_S^{(\chi)} a_{\psi 1}}{8r^2} + \frac{Z}{8r^4} (a_{\chi 1} - 4a_{\chi 4}), \quad \frac{\delta d_{13}}{g^2} = -\frac{\delta d_{15}}{g^2} = \frac{c_F^{(\chi)} a_{\psi 4}}{2r} + \frac{c_F a_{\chi 4}}{2r^3}.\end{aligned}\quad (33)$$

The coefficient relations (7), (9) and (25) are preserved, since by construction the Lorentz transformation properties of  $A^\mu$  are unchanged and hence the boost transformation rules (3) and (4) still apply.

We may use (31) to eliminate vacuum polarization terms  $c_{V2}$  and  $c_{V4}$  in favor of compensating terms in (32). Similarly, (32), together with the analogous relations for  $c_i^{(\chi)}$ , and (33), can be used to eliminate 10 linear combinations of Wilson coefficients for two-fermion and four-fermion operators. Different applications may favor elimination of different operators.<sup>9</sup>

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<sup>9</sup> We have not specified gauge fixing and source terms, which are also affected by field redefinitions.

## 5.4 Relativistic lepton

For applications such as lepton-nucleon scattering at energies  $m_\ell, E \ll M$  (e.g., low-energy electron-proton scattering), the relevant effective theory involves a heavy fermion (e.g., the proton) interacting with an electromagnetically charged relativistic fermion (e.g., the electron). Let us briefly discuss this case. Enforcing parity, time-reversal, gauge, Lorentz, as well as chiral symmetry at  $m_\ell = 0$ , we find the leptonic interactions with the photon,

$$\mathcal{L}_\ell = \bar{\ell} \left[ i\not{D} - m_\ell + gc_F^{(\ell)} m_\ell \frac{\sigma^{\mu\nu} F_{\mu\nu}}{M^2} + gc_2^{(\ell)} m_\ell \frac{D^2}{M^2} + gc_D^{(\ell)} \frac{[\partial^\mu F_{\mu\nu}] \gamma^\nu}{M^2} + \mathcal{O}(1/M^4) \right] \ell, \quad (34)$$

where we assume field redefinitions have been performed to remove power suppressed terms involving  $(i\not{D} - m_\ell)\ell$ .

Having performed field redefinitions to eliminate operators with time derivatives acting on fermion fields, the lagrangian for the nucleon-relativistic lepton sector through  $\mathcal{O}(1/M^3)$  is

$$\begin{aligned} \mathcal{L}_{\psi\ell} = & \frac{b_1}{M^2} \psi^\dagger \psi \bar{\ell} \gamma^0 \ell + \frac{b_2}{M^2} \psi^\dagger \sigma^i \psi \bar{\ell} \gamma^i \gamma_5 \ell + \frac{b_3}{M^3} \psi^\dagger \psi m_\ell \bar{\ell} \ell + \frac{b_4}{M^3} \psi^\dagger i D_-^i \psi \bar{\ell} \gamma^i \ell \\ & + \frac{b_5}{M^3} \psi^\dagger \psi \bar{\ell} i \boldsymbol{\gamma} \cdot \mathbf{D}_- \ell + \frac{b_6}{M^3} \epsilon^{ijk} \psi^\dagger \sigma^i \psi m_\ell \bar{\ell} \sigma^{jk} \ell + \frac{b_7}{M^3} \epsilon^{ijk} \psi^\dagger \sigma^i \psi \bar{\ell} \gamma^j D_+^k \ell \\ & + \frac{b_8}{M^3} \psi^\dagger \sigma^i \psi \bar{\ell} \gamma^0 \gamma_5 i D_-^i \ell + \frac{b_9}{M^3} \psi^\dagger \sigma^i i D_-^i \psi \bar{\ell} \gamma^0 \gamma_5 \ell + \mathcal{O}(1/M^4), \end{aligned} \quad (35)$$

where  $\ell$  is the relativistic lepton field with mass  $m_\ell$  and  $\sigma^{ij} \equiv \frac{i}{2}[\gamma^i, \gamma^j]$ . The heavy field  $\psi$  transforms under boosts as in (3), while  $\ell$  transforms under finite dimensional representations of the Lorentz group in the usual way. Under Lorentz transformation, we thus find

$$\delta \mathcal{L}_{\psi\ell} = -\frac{1}{M^3} \psi^\dagger \psi \bar{\ell} q^i \gamma^i \ell (b_1 + 2b_4) - \frac{1}{M^3} \psi^\dagger \sigma^i q^i \psi \bar{\ell} \gamma^0 \gamma_5 \ell (b_2 + 2b_9) + \mathcal{O}(1/M^4). \quad (36)$$

This enforces the relations

$$b_4 = \frac{1}{2} b_1, \quad b_9 = -\frac{1}{2} b_2, \quad (37)$$

leaving seven operators in this sector through order  $1/M^3$ , including two at order  $1/M^2$ .

By performing field redefinitions on the gauge field  $A^\mu$ , some of these four-fermion operators are found to mix with one-heavy particle operators. In addition to the contributions  $\Delta_\gamma A^\mu$  and  $\Delta_\psi A^\mu$  from (27) we may employ

$$\Delta_\ell A^\mu = ga_{\ell 1} \frac{\bar{\ell} \gamma^\mu \ell}{M^2} + \mathcal{O}(1/M^4). \quad (38)$$

We thus find the modified couplings in  $\mathcal{L}_\ell$ ,

$$\delta c_D^{(\ell)} = -Z_\ell a_{\gamma 1} + a_{\ell 1}, \quad (39)$$

and for the four fermion operators in  $\mathcal{L}_{\psi\ell}$ ,

$$\frac{\delta b_1}{g^2} = -Z a_{\ell 1} - Z_\ell a_{\psi 1}, \quad \frac{\delta b_7}{g^2} = -Z_\ell a_{\psi 4} - \frac{1}{2} c_F a_{\ell 1}, \quad (40)$$

with relation (37) remaining intact.

## 6 Applications

Applications of the NRQED lagrangian for the nucleon include: the computation of proton structure effects in atomic bound states; the model-independent analysis of radiative corrections to low-energy lepton-nucleon scattering; and the study of static electromagnetic properties of nucleons. Let us discuss sample applications in each of these areas.

### 6.1 Proton structure in atomic bound states

As a sample computation involving the  $1/M^4$  NRQED lagrangian,<sup>10</sup> let us analyze the effects of nuclear structure on the two-photon exchange contribution to the  $2S-2P$  Lamb shift in hydrogenic bound states; this contribution is the subject of intense scrutiny [16, 17, 18, 19, 3, 20] due to the discrepant measurements in muonic and electronic hydrogen [2]. The dominant theoretical uncertainties in the muonic hydrogen Lamb shift arise from proton structure, represented by contributions through  $\mathcal{O}(\alpha)$  to  $c_D$  in (1), arising from first order vertex corrections, and  $\mathcal{O}(\alpha^2)$  contributions to  $d_2$  in (23) arising from two-photon exchange [3]. We let  $\psi$  of Section 2 denote the proton of mass  $M$ , and  $\chi$  of Section 5 denote the electron (or muon) of mass  $M_\chi = m_e$ . Electric charge assignments are given after (1). We focus here on the model-independent result for the leading two-photon exchange corrections in the limit  $m_e/M \ll 1$ .

Structure dependent corrections to  $d_2$  lead to first order energy shifts in hydrogen,<sup>11</sup>

$$\delta E(n, \ell) = \delta_{\ell 0} \frac{m_r^3 \alpha^3}{\pi n^3} \left( -\frac{\delta d_2}{M^2} \right), \quad (41)$$

where  $(n, \ell)$  are principal and orbital quantum numbers and  $m_r = m_e M / (m_e + M)$  is the reduced mass. The matching condition for  $d_2$  involves a weighted moment of the structure functions for forward Compton scattering on the proton. From [3], the necessary hadronic input is contained in

$$\frac{\delta d_2}{M^2} = -\frac{4m_e \alpha^2}{\pi M} \int_{-1}^1 dx \sqrt{1-x^2} \int_0^\infty \frac{dQ}{Q} \frac{[(1+2x^2)W_1 - (1-x^2)m_p^2 W_2]}{Q^2 + 4m_e^2 x^2} + \text{subtractions}, \quad (42)$$

where the subtractions depend on infrared and ultraviolet regulators. The structure functions in this expression are evaluated at  $\nu^2 = -4x^2 M^2 Q^2$ . They are defined by the forward proton matrix element,

$$W^{\mu\nu}(k, q, s) \equiv i \int d^4x e^{iq \cdot x} \langle \mathbf{k}, s | T \{ J_{\text{e.m.}}^\mu(x) J_{\text{e.m.}}^\nu(0) \} | \mathbf{k}, s \rangle. \quad (43)$$

---

<sup>10</sup> For bound state applications it may be computationally efficient to further integrate out offshell momentum modes and/or higher Fock states to arrive at an explicitly  $v/c$  expanded NRQED [1, 9], or fixed particle number quantum mechanics [15]. Our purpose here is to examine the impact of nucleon structure, as described by finite shifts in the Wilson coefficients of the theory described by (1), (22), (23).

<sup>11</sup> As mentioned in Section 5.2, we have used powers of  $1/M$  (not  $1/M_\chi$ ) to define four-fermion Wilson coefficients in (23). We thus have  $d_{1,2}/M^2 = d_{1,2}^{\text{CL}}/(m_e M)$  where  $d_2^{\text{CL}}$  is the coefficient of Caswell and Lepage [1], also employed in [3].

For the present application we require the spin-averaged component symmetric in  $\mu, \nu$ ,

$$W_S^{\mu\nu} = \frac{1}{2} \sum_s W^{\mu\nu} = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) W_1(\nu, Q^2) + \left( k^\mu - \frac{k \cdot q q^\mu}{q^2} \right) \left( k^\nu - \frac{k \cdot q q^\nu}{q^2} \right) W_2(\nu, Q^2), \quad (44)$$

where  $Q^2 = -q^2$  and  $\nu = 2k \cdot q$ . Our normalizations are such that for a point particle,  $W_1 = 2\nu^2/(Q^4 - \nu^2)$  and  $W_2 = 8Q^2/(Q^4 - \nu^2)$ .

Let us proceed to evaluate the finite shift  $\delta d_2$  due to proton structure, and hence  $\delta E$  in (41), in the limit  $m_e \ll M$ . Details of this computation are presented elsewhere. The essential input from NRQED is the evaluation of  $W_i(0, Q^2)$  through  $\mathcal{O}(Q^2)$ ,

$$\begin{aligned} W_1(0, Q^2) &= -2 + 2c_F^2 + \frac{Q^2}{2M^2} (c_F^2 - 2c_F c_{W1} + 2c_M + c_{A1}) + \mathcal{O}(Q^4), \\ M^2 W_2(0, Q^2) &= \frac{8M^2}{Q^2} + 2c_F^2 - 2c_D + \frac{Q^2}{2M^2} \left[ -1 + c_F^2 - c_D + \frac{1}{4}c_D^2 + 2c_M - 2c_F c_{W1} - \frac{1}{2}c_{A2} \right. \\ &\quad \left. + 32(c_{X1} + c_{X2} + c_{X3}) \right] + \mathcal{O}(Q^4), \end{aligned} \quad (45)$$

and the evaluation of  $W_1(\nu, 0)$  through  $\mathcal{O}(\nu^2)$ ,

$$\begin{aligned} W_1(\nu, 0) &= -2 + \frac{\nu^2}{8M^4} \left( -c_F^2 + c_F c_S + 2c_M - \frac{1}{2}c_{A2} \right) + \mathcal{O}(\nu^4), \\ \lim_{Q^2 \rightarrow 0} W_2(\nu, Q^2) &= 0, \end{aligned} \quad (46)$$

where the final relation follows from requiring that the Compton amplitude (44) is finite at  $Q^2 \rightarrow 0$ . Note in particular that  $W_2(0, Q^2)$  relies on  $\mathcal{O}(1/M^4)$  one-photon vertices involving  $c_{X1}$ ,  $c_{X2}$ ,  $c_{X3}$ . This occurs because the  $\mathcal{O}(Q^2)$  terms in  $W_2(0, Q^2)$  arise at third order in the  $Q^2/M^2$  expansion. Employing these results in the matching relation (42) yields the leading contribution to  $S$ -state energies at  $m_e/M \rightarrow 0$ ,

$$\delta E(nS) = \frac{m_e^3 \alpha^5}{\pi n^3} \frac{m_e}{M^3} \left\{ \log \frac{m_e}{M} \left[ M^3 (5\alpha_E - \beta_M) / \alpha - 3a_p(1 + a_p) + 2M^2(r_E^p)^2 \right] + \dots \right\}. \quad (47)$$

Our result for the coefficient of the logarithm containing  $\alpha_E$  and  $\beta_M$  was obtained in a temporal gauge analysis in [21]. We have here used NRQED to establish the complete logarithmically enhanced contribution. The expansion at small lepton mass is not possible for muonic hydrogen owing to low hadronic scales  $m_\mu \sim m_\pi \sim m_\Delta - m_p$ . Further analysis for this case will be presented elsewhere.

## 6.2 Radiative corrections to lepton-nucleon scattering

Extraction of key hadronic quantities from elastic lepton-nucleon scattering, such as the charge and magnetic radii of the proton, demands rigorous control over radiative corrections. Traditional analyses of the two-photon exchange contribution to elastic electron-proton scattering



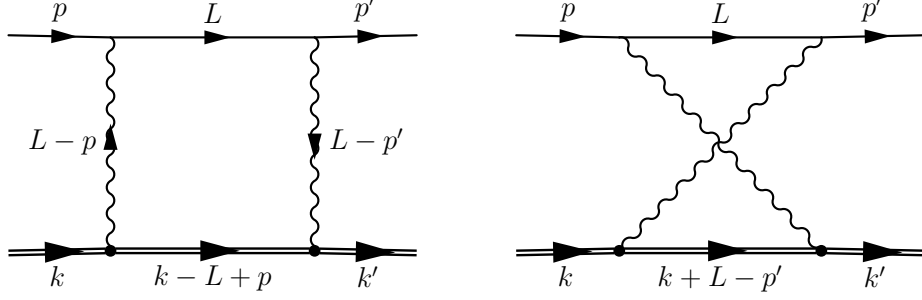


Figure 3: Diagrams contributing at leading order in  $1/M$  to two-photon exchange in electron-proton scattering.

resort to hadronic models involving phenomenological form factor insertions into point-particle Feynman diagrams [22, 23, 24, 25], or require modeling of soft functions to evaluate expressions obtained in a hard-scattering factorization framework [26]. At low energies,  $E \ll M$ , a systematic and model-independent approach is provided by NRQED. This region has overlap with current and planned electron-proton and muon-proton scattering measurements, and provides a rigorous test of hadronic models employed at higher energy.

Consider the two-photon exchange corrections to low-energy electron-proton scattering in the limit  $m_\ell = m_e \ll E \ll M$ , where  $E$  is the energy of the incident electron in the rest frame of the initial-state proton. The appropriate effective theory consists of a relativistic lepton and heavy proton, as described in Section 5.4. The leading order NRQED diagrams are not sensitive to proton structure, yielding, in Feynman gauge, [electric charge assignments are given after (1)]

$$i\mathcal{M} \approx e^4 \bar{u}(p') \gamma^0 \gamma^\mu \gamma^0 u(p) \times \int \frac{d^4 L}{(2\pi)^4} \frac{L_\mu}{L^2 + i0} \frac{1}{(L-p)^2 - \lambda^2 + i0} \frac{1}{(L-p')^2 - \lambda^2 + i0} \left( \frac{1}{-L^0 + E + i0} + \frac{1}{L^0 - E + i0} \right), \quad (48)$$

where we have neglected the electron mass and used that  $p^0 - p'^0$ ,  $k^0 - M$ , and  $k'^0 - M$  are subleading in the  $1/M$  expansion to simplify the integrand. A photon mass  $\lambda$  is used to regulate infrared divergences. Evaluating the integral in the limit  $\lambda \ll E$ , we recover the expression for relativistic lepton scattering in an external Coulomb field [27, 28],<sup>12</sup>

$$\mathcal{M} = \frac{4\pi\alpha}{Q^2} \bar{u}(p) \gamma^0 u(p) \left\{ 1 + \alpha \left[ \frac{\pi}{2} \frac{Q}{2E + Q} + i \left( -2 \log \frac{\lambda}{Q} + \frac{Q^2}{(2E)^2 - Q^2} \log \frac{Q}{2E} \right) \right] + \mathcal{O}[\alpha^2, \lambda/E, m_e/E, E/M] \right\}. \quad (49)$$

<sup>12</sup> There is an apparent typographical error in Eq. (2.6) of [28] where in the second line the integral  $I$  should have the opposite overall sign. This affects only the imaginary part of the  $\mathcal{O}(\alpha)$  correction to the leading amplitude and is thus not relevant to first order radiative corrections in the cross section.

The power of the effective theory lies in the possibility to systematically compute corrections in powers of  $E/M$  and to relate observables such as scattering amplitudes and bound state energies, in a model-independent fashion. Sensitivity to the Wilson coefficient  $c_F$  appears at order  $1/M$  in the proton spin-dependent amplitude. At order  $1/M^2$  there is a dependence on  $c_D$ , and on the constants  $b_1$  and  $b_2$  from (35), which encode information on proton excitations and may be related to moments of the forward Compton amplitude.

A similar analysis may be performed to compute radiative corrections to muon-proton scattering in the limit  $E \ll m_\mu \sim M$ , where now the effective theory consists of heavy muon and proton fields, as discussed in Section 5.2. Proton structure not captured by elastic form factors is first encountered at  $\mathcal{O}(1/M^2)$ , encoded in coefficients  $d_1$  and  $d_2$  in (23). NRQED may similarly be used to compute radiative backgrounds to searches for possible new low-mass and weakly coupled particles using low-energy electron-proton scattering [29].

### 6.3 Spin polarizabilities and static properties of nucleons

Having constructed the NRQED lagrangian, it is straightforward to relate parameters extracted from scattering measurements to those determined by static nucleon properties. Consider for simplicity the case of a neutral particle such as the neutron.<sup>13</sup> For example, the shift in energy due to an external electromagnetic field is determined by the zero-momentum limit of the amputated two-point function,

$$\begin{aligned} -\delta M(\mathbf{E}, \mathbf{B}) &= \lim_{\mathbf{p} \rightarrow 0} \Sigma(\mathbf{p}) \\ &= c_F g \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2M} + c_D g \frac{[\boldsymbol{\partial} \cdot \mathbf{E}]}{8M^2} + c_{A1} g^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8M^3} - c_{A2} g^2 \frac{\mathbf{E}^2}{16M^3} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial} \cdot \mathbf{E}]}{M^4} \\ &\quad + c_{X7} g^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{B} [\boldsymbol{\partial} \cdot \mathbf{E}]}{M^4} + c_{X8} g^2 \frac{[\mathbf{E} \cdot \boldsymbol{\partial} \boldsymbol{\sigma} \cdot \mathbf{B}]}{M^4} + c_{X9} g^2 \frac{[\mathbf{B} \cdot \boldsymbol{\partial} \boldsymbol{\sigma} \cdot \mathbf{E}]}{M^4} \\ &\quad + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} E^i]}{M^4} - c_{X12} g^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{E} \times [\boldsymbol{\partial} \times \mathbf{B}]}{M^4}, \end{aligned} \quad (50)$$

and we may immediately identify the various static electromagnetic moments. The identification of  $c_{X_i}$  as coefficients of a well-defined effective theory is essential for correctly relating static coefficients, as measured by lattice [31, 32, 33, 34] or hadronic models, to scattering observables [35, 36]. For example, conventional definitions of the scalar electric ( $\alpha_E$ ) and magnetic ( $\beta_M$ ) polarizabilities as determined by Compton scattering [37] require that coefficients  $c_{A1}$  and  $c_{A2}$  are not proportional to these quantities, but rather<sup>14</sup>

$$\begin{aligned} \frac{4M^3}{\alpha} \alpha_E &= -c_{A1} - \frac{1}{2} c_{A2} + Z^2 + 2Zc_M + c_F c_S - c_F^2, \\ \frac{4M^3}{\alpha} \beta_M &= c_{A1} - Z^2. \end{aligned} \quad (51)$$

<sup>13</sup> Static properties of a charged particle require care in definition, see, e.g., [30].

<sup>14</sup> Many approaches in the literature lack a systematic treatment of these effects [17, 35, 33, 34, 36]. The NRQED lagrangian, constrained by Lorentz invariance, trivializes relations such as (51) between static nucleon properties and scattering observables.

Similar relations relate static spin polarizabilities to scattering observables: see (21) and the associated footnote.

The coefficients of spin-dependent operators  $c_{Xi}$  contribute to the spin-dependent structure functions of forward Compton scattering, obtained from the antisymmetric component of (43),

$$W_A^{\mu\nu} = \bar{u}(k) \left\{ H_1 \left( [\gamma^\mu, \not{q}] k^\nu - [\gamma^\nu, \not{q}] k^\mu - [\gamma^\mu, \gamma^\nu] k \cdot q \right) + H_2 \left( [\gamma^\mu, \not{q}] q^\nu - [\gamma^\nu, \not{q}] q^\mu - [\gamma^\mu, \gamma^\nu] q^2 \right) \right\} u(k), \quad (52)$$

where our normalization conventions are such that for a point particle  $MH_1 = Q^2/(Q^4 - \nu^2)$ ,  $H_2 = 0$ . The model-independent analysis of these functions impacts structure-dependent corrections in hydrogen spectroscopy.<sup>15</sup> Let us compute  $H_1(0, Q^2)$  by using NRQED to compute  $W_A^{i0}$ . This requires a subclass of  $1/M^5$  interactions involving the magnetic field and four spatial derivatives,

$$\begin{aligned} \Delta\mathcal{L} = \frac{g}{M^5} \psi^\dagger \bigg\{ & c_{Y1} \{ \boldsymbol{\sigma} \cdot \mathbf{B}, \partial^4 \} + c_{Y2} \{ \partial^2, \partial^i \boldsymbol{\sigma} \cdot \mathbf{B} \partial^i \} + c_{Y3} \partial^2 \boldsymbol{\sigma} \cdot \mathbf{B} \partial^2 + c_{Y4} \partial^i \partial^j \boldsymbol{\sigma} \cdot \mathbf{B} \partial^i \partial^j \\ & + c_{Y5} (\mathbf{B} \cdot \partial \boldsymbol{\sigma} \cdot \partial \partial^2 + \partial^2 \boldsymbol{\sigma} \cdot \partial \partial \cdot \mathbf{B}) + c_{Y6} (\boldsymbol{\sigma} \cdot \partial \mathbf{B} \cdot \partial \partial^2 + \partial^2 \partial \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \partial) \\ & + c_{Y7} (\partial^i B^j \boldsymbol{\sigma} \cdot \partial \partial^i \partial^j + \partial^i \partial^j \boldsymbol{\sigma} \cdot \partial B^j \partial^i) + \mathcal{O}(g) \bigg\} \psi, \end{aligned} \quad (53)$$

where we have neglected effects in setting  $\mathbf{D} \rightarrow \boldsymbol{\partial}$  that are not relevant to the present application. At low  $Q^2$  we find

$$\begin{aligned} 2MH_1(0, Q^2) = & \frac{2}{Q^2} Z c_F + \frac{1}{4M^2} \left[ Z(2c_F + c_S - 2c_{W1}) - c_F c_D \right] \\ & + \frac{Q^2}{16M^4} \left[ c_D(c_{W1} - c_F) - 2Z(c_F + c_{W1} - 32c_{Y1}) \right. \\ & \left. + 32c_F(c_{X1} + c_{X2} + c_{X3}) + 16(-c_{X7} + c_{X8} + c_{X12}) \right] + \mathcal{O}(Q^4). \end{aligned} \quad (54)$$

Using relations (15), (16), (21), and

$$c_{Y1} = \frac{27}{256} \bar{F}_1 + \frac{23}{256} \bar{F}_2 + \frac{5}{16} \bar{F}_1' + \frac{5}{16} \bar{F}_2' + \frac{1}{4} \bar{F}_1'' + \frac{1}{4} \bar{F}_2'', \quad (55)$$

the result (54) may be expressed in terms of scattering observables.

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<sup>15</sup> These functions enter directly in the analysis of spin-dependent transitions. Additionally, ansatzes used to model  $W_1(0, Q^2)$  may be compared to analogous studies of  $H_1(0, Q^2)$ , which by virtue of an unsubtracted dispersion relation, can be reconstructed from experimental data on elastic and inelastic scattering [38].

## 7 Summary

We have derived a complete basis of operators and coefficient constraints through order  $1/M^4$  for the parity and time-reversal invariant effective lagrangian for a heavy fermion interacting with an abelian gauge field, i.e., NRQED.

The computations of this paper provide an illustration of relativity constraints in high orders of heavy particle effective theories [8]. In particular, the transformation law for fields under Lorentz boosts receives nontrivial corrections compared to a naive reparameterization ansatz [11]. These effects occur first in the  $1/M^4$  lagrangian, and are validated by the explicit matching and variational computations presented in this paper. Relations (9) and (25), and their extensions to QCD (e.g. HQET or NRQCD) represent new non-renormalization theorems valid to all orders in perturbation theory.

NRQED can be used to efficiently analyze processes involving long wavelength electromagnetic probes of the nucleon. The analysis was motivated in part by the necessity to incorporate high-order corrections of proton structure in hydrogenic bound states. A sample application to structure-dependent two-photon exchange corrections in the small lepton mass limit was presented in Section 6.1.

For applications to low-energy electron scattering,  $m_e \ll E \ll m_p$ , we constructed in Section 5.4 the effective theory for a relativistic electron and nonrelativistic proton. The lagrangian and coefficient relations were derived through  $\mathcal{O}(1/M^3)$ . This case may be applied to a model-independent analysis of radiative corrections in the extraction of proton structure from electron scattering, as discussed in Section 6.2.

The effects of nucleon structure are implemented by non-pointlike values for the Wilson coefficients appearing in (1). Having determined the structure of the NRQED lagrangian, it becomes a trivial task to define and compute static properties of nucleons, and unambiguously relate these properties to scattering measurements, as illustrated in Section 6.3.

For simplicity we focused in this paper on the effective lagrangian for a parity conserving theory of a heavy fermion coupled to an abelian gauge field. Extensions to nonabelian gauge fields, the inclusion of parity violation, and the consideration of different heavy particle spins each have important applications.

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